

Statistical mechanics and thermodynamic limit of self-gravitating fermions in D dimensions

Pierre-Henri Chavanis

Laboratoire de Physique Théorique, Université Paul Sabatier,
118 route de Narbonne 31062 Toulouse, France.

Abstract

We discuss the statistical mechanics of a system of self-gravitating fermions in a space of dimension D . We plot the caloric curves of the self-gravitating Fermi gas giving the temperature as a function of energy and investigate the nature of phase transitions as a function of the dimension of space. We consider stable states (global entropy maxima) as well as metastable states (local entropy maxima). We show that for $D \geq 4$, there exists a critical temperature (for sufficiently large systems) and a critical energy below which the system cannot be found in statistical equilibrium. Therefore, for $D \geq 4$, quantum mechanics cannot stabilize matter against gravitational collapse. This is similar to a result found by Ehrenfest (1917) at the atomic level for Coulombian forces. This makes the dimension $D = 3$ of our universe very particular with possible implications regarding the anthropic principle. Our study enters in a long tradition of scientific and philosophical papers who studied how the dimension of space affects the laws of physics.

1 Introduction

The statistical mechanics of systems with long-range interactions is currently a topic of active research [1]. Among long-range interactions, the gravitational force plays a fundamental role. Therefore, the development of a statistical mechanics for self-gravitating systems is of considerable interest [2]. In this context, a system of self-gravitating fermions enclosed within a box provides an interesting model which can be studied in great detail [3, 4]. This model incorporates an effective small-scale cut-off played by the Pauli exclusion principle and a large scale cut-off played by the confining box (other forms of confinement could also be considered). The statistical mechanics of this system is rigorously justified and presents a lot of interesting features which are of interest in statistical mechanics [5] and astrophysics [6]. Its detailed study is therefore important at a conceptual and practical level.

In a preceding paper [4], we have discussed the nature of phase transitions in the self-gravitating Fermi gas in a space of dimension $D = 3$. Our study was performed in both microcanonical and canonical ensembles and considered an arbitrary degree of degeneracy relative to the system size. This study completes previous investigations by Hertel & Thirring [3] who worked in the canonical ensemble and considered small system sizes. At high temperatures and high energies, the system is in a gaseous phase and quantum effects are completely negligible. At some transition temperature T_t or transition energy E_t (for sufficiently large system sizes), a first order phase transition is expected to occur and drive the system towards a condensed phase. However, gaseous states are still metastable below this transition point

and gravitational collapse will rather occur at a smaller critical temperature T_c (Jeans temperature) [7] or critical energy E_c (Antonov energy) [8, 9, 2] at which the metastable branch disappears (spinodal point). The end-state of the collapse is a compact object with a “core-halo” structure. Typically, it consists of a degenerate nucleus surrounded by a “vapour”. The nucleus (condensate) resembles a white dwarf star [10]. At non-zero temperature, this compact object is surrounded by a dilute atmosphere. Therefore, when quantum mechanics is properly accounted for, there exists an equilibrium state (global maximum of entropy or free energy) for each value of accessible energy and temperature. The condensate results from the balance between gravitational contraction and quantum pressure. As first noticed by Fowler [11] in his classical theory of white dwarf stars, quantum mechanics is able to stabilize matter against gravitational collapse.

One object of this paper is to show that this conclusion is no more valid in a space of dimension $D \geq 4$. For a system of mass M enclosed within a box of radius R , there exists a critical temperature (for sufficiently large R) and a critical energy below which the system cannot be found at statistical equilibrium. This is like the Antonov instability for self-gravitating classical particles in $D = 3$ [8, 9, 2] but it now occurs for fermions. Therefore, quantum mechanics cannot arrest gravitational collapse in $D \geq 4$. This result is connected to our previous observation [12] that a classical white dwarf star (a polytrope of index $n_{3/2} = D/2$) becomes unstable for $D \geq 4$. Interestingly, this result is similar to that of Ehrenfest [13] who considered the stability of atomic structures (in Bohr’s model) for different dimensions of space and concludes that $D < 4$ is required for stability. In this paper, we determine the caloric curve of the self-gravitating Fermi gas for an arbitrary dimension of space and an arbitrary degree of degeneracy (or system size). We exhibit particular dimensions that play a special role in the problem. The dimension $D = 2$ is critical because the results established for $D \neq 2$ cannot be directly extended to $D = 2$ [14]. Furthermore, in $D = 2$ the radius of a white dwarf star is independent on its mass and given in terms of fundamental constants by $R = 0.27 \, h m^{-3/2} G^{-1/2}$. The dimension $D = 4$ is also critical because it is the dimension at which classical white dwarf stars become unstable. At this particular dimension, their mass is independent on radius and can be expressed in terms of fundamental constants as $M = 1.44 \, 10^{-2} h^4 m^{-5} G^{-2}$. Mathematically, this is similar to Chandrasekhar’s limiting mass [15] for relativistic white dwarf stars in $D = 3$. The dimension $D = 2(1 + \sqrt{2})$ is also particular because at this dimension, the white dwarf stars cease to be self-confined and have infinite mass. Finally, $D = 10$ is the dimension at which the caloric curve of classical isothermal spheres loses its characteristic spiral nature [14]. Although we systematically explore all dimensions of space in order to have a complete picture of the problem, only dimensions $D = 1$, $D = 2$ and $D = 3$ are *a priori* of physical interest. The dimension $D = 1$ is considered in cosmology and in connexion with shell models, and the dimension $D = 2$ can be useful to describe filaments or ring structures with high aspect ratio. Two-dimensional gravity is also of interest for its properties of conformal invariance and for its relation with two-dimensional turbulence [5]. Non-integer dimensions can arise if the system has a fractal nature.

The paper is organized as follows. In Sec. 2, we determine the thermodynamical parameters of the self-gravitating Fermi gas in dimension D . The Fermi-Dirac entropy is introduced from a combinatorial analysis. In Sec. 3, we consider asymptotic limits corresponding to the classical self-gravitating gas and to completely degenerate structures (white dwarfs). We emphasize the importance of metastable states in astrophysics and explain how they can be taken into account in the theory (see also [16]). We also discuss the thermodynamic limit of the self-gravitating quantum gas and compare it with the thermodynamic limit of the self-gravitating classical gas in the dilute limit [17]. In Sec. 4, we provide a gallery of caloric curves of the self-gravitating Fermi gas in different dimensions of space. Rigorous mathematical results on the existence of

solutions of the Fermi-Poisson equation have been obtained by Stańczy [18]. Finally, in the conclusion, we place our study in a more general perspective. We give a short historical account of scientific and philosophical papers who studied the role played by the dimension of space in determining the form of the laws of physics. These works tend to indicate that the dimension $D = 3$ of our universe is very particular. This is also the result that we reach in our study. These remarks can have implications regarding the anthropic principle.

2 Thermodynamics of self-gravitating D -fermions

2.1 The Fermi-Dirac distribution

We consider a system of N fermions interacting via Newtonian gravity in a space of dimension D . We assume that the mass of the configuration is sufficiently small so as to ignore relativistic effects. Let $f(\mathbf{r}, \mathbf{v}, t)$ denote the distribution function of the system, i.e. $f(\mathbf{r}, \mathbf{v}, t)d^D\mathbf{r}d^D\mathbf{v}$ gives the mass of particles whose position and velocity are in the cell $(\mathbf{r}, \mathbf{v}; \mathbf{r} + d^D\mathbf{r}, \mathbf{v} + d^D\mathbf{v})$ at time t . The integral of f over the velocity determines the spatial density

$$\rho = \int f d^D\mathbf{v}, \quad (1)$$

and the total mass of the configuration is given by

$$M = \int \rho d^D\mathbf{r}, \quad (2)$$

where the integral extends over the entire domain. On the other hand, in the meanfield approximation, the total energy of the system can be expressed as

$$E = \frac{1}{2} \int f v^2 d^D\mathbf{r} d^D\mathbf{v} + \frac{1}{2} \int \rho \Phi d^D\mathbf{r} = K + W, \quad (3)$$

where K is the kinetic energy and W the potential energy. The gravitational potential Φ is related to the density by the Newton-Poisson equation

$$\Delta\Phi = S_D G \rho, \quad (4)$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface of a unit sphere in a space of dimension D and G is the constant of gravity (which depends on the dimension of space).

We now wish to determine the most probable distribution of self-gravitating fermions at statistical equilibrium. To that purpose, we divide the individual phase space $\{\mathbf{r}, \mathbf{v}\}$ into a very large number of microcells with size $(h/m)^D$ where h is the Planck constant (the mass m of the particles arises because we use \mathbf{v} instead of \mathbf{p} as a phase space coordinate). A microcell is occupied either by 0 or 1 fermion (or $g = 2s + 1$ fermions if we account for the spin). We shall now group these microcells into macrocells each of which contains many microcells but remains nevertheless small compared to the phase-space extension of the whole system. We call ν the number of microcells in a macrocell. Consider the configuration $\{n_i\}$ where there are n_1 fermions in the 1st macrocell, n_2 in the 2nd macrocell etc..., each occupying one of the ν microcells with no cohabitation. The number of ways of assigning a microcell to the first element of a macrocell is ν , to the second $\nu - 1$ etc. Since the particles are indistinguishable, the number of ways of assigning microcells to all n_i particles in a macrocell is thus

$$\frac{1}{n_i!} \times \frac{\nu!}{(\nu - n_i)!}. \quad (5)$$

To obtain the number of microstates corresponding to the macrostate $\{n_i\}$ defined by the number of fermions n_i in each macrocell (irrespective of their precise position in the cell), we need to take the product of terms such as (5) over all macrocells. Thus, the number of microstates corresponding to the macrostate $\{n_i\}$, i.e. the probability of the state $\{n_i\}$, is

$$W(\{n_i\}) = \prod_i \frac{\nu!}{n_i!(\nu - n_i)!}. \quad (6)$$

This is the Fermi-Dirac statistics. As is customary, we define the entropy of the state $\{n_i\}$ by

$$S(\{n_i\}) = \ln W(\{n_i\}). \quad (7)$$

It is convenient here to return to a representation in terms of the distribution function giving the phase-space density in the i -th macrocell

$$f_i = f(\mathbf{r}_i, \mathbf{v}_i) = \frac{n_i}{\nu} \frac{m}{(\frac{h}{m})^D} = \frac{n_i \eta_0}{\nu}, \quad (8)$$

where we have defined $\eta_0 = m^{D+1}/h^D$, which represents the maximum value of f due to Pauli's exclusion principle. Now, using the Stirling formula, we have

$$\ln W(\{n_i\}) \simeq \sum_i \nu (\ln \nu - 1) - \nu \left\{ \frac{f_i}{\eta_0} \left[\ln \left(\frac{\nu f_i}{\eta_0} \right) - 1 \right] + \left(1 - \frac{f_i}{\eta_0} \right) \left[\ln \left\{ \nu \left(1 - \frac{f_i}{\eta_0} \right) \right\} - 1 \right] \right\}. \quad (9)$$

Passing to the continuum limit $\nu \rightarrow 0$, we obtain the usual expression of the Fermi-Dirac entropy

$$S = -k_B \int \left\{ \frac{f}{\eta_0} \ln \frac{f}{\eta_0} + \left(1 - \frac{f}{\eta_0} \right) \ln \left(1 - \frac{f}{\eta_0} \right) \right\} \frac{d^D \mathbf{r} d^D \mathbf{v}}{(\frac{h}{m})^D}. \quad (10)$$

If we take into account the spin of the particles, the above expression remains valid but the maximum value of the distribution function is now $\eta_0 = gm^{D+1}/h^D$, where $g = 2s + 1$ is the spin multiplicity of the quantum states (the phase space element has also to be multiplied by g). An expression of entropy similar to (10), but arising for a completely different reason, has been introduced by Lynden-Bell in the context of the violent relaxation of collisionless stellar systems [19, 20, 21]. In that context, η_0 represents the maximum value of the initial distribution function and the actual distribution function (coarse-grained) must always satisfy $\bar{f} \leq \eta_0$ by virtue of the Liouville theorem. This is the origin of the ‘‘effective’’ exclusion principle in Lynden-Bell's theory, which has nothing to do with quantum mechanics. Since the particles (stars) are distinguishable classical objects (but subject to an exclusion principle in the collisionless regime), Lynden-Bell's statistics corresponds to a 4-th form of statistics (in addition to the Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein statistics). However, for a single type of phase element η_0 , it leads to the same results as the Fermi-Dirac statistics. We also recall that in the non-degenerate (or classical) limit $f \ll \eta_0$, the Fermi-Dirac entropy (10) reduces to the Boltzmann entropy

$$S = -k_B \int \frac{f}{m} \left[\ln \left(\frac{f h^D}{g m^{D+1}} \right) - 1 \right] d^D \mathbf{r} d^D \mathbf{v}. \quad (11)$$

Now that the entropy has been precisely justified, the statistical equilibrium state (most probable state) of self-gravitating fermions is obtained by maximizing the Fermi-Dirac entropy (10) at fixed mass (2) and energy (3):

$$\text{Max } S[f] \quad | \quad E[f] = E, M[f] = M. \quad (12)$$

Introducing Lagrange multipliers $1/T$ (inverse temperature) and μ (chemical potential) to satisfy these constraints, and writing the variational principle in the form

$$\delta S - \frac{1}{T} \delta E + \frac{\mu}{T} \delta N = 0, \quad (13)$$

we find that the *critical points* of entropy correspond to the Fermi-Dirac distribution

$$f = \frac{\eta_0}{1 + \lambda e^{\beta m(\frac{v^2}{2} + \Phi)}}, \quad (14)$$

where $\lambda = e^{-\beta\mu}$ is a strictly positive constant (inverse fugacity) and $\beta = \frac{1}{k_B T}$ is the inverse temperature. Clearly, the distribution function satisfies $f \leq \eta_0$, which is a consequence of Pauli's exclusion principle.

So far, we have assumed that the system is isolated so that the energy is conserved. If now the system is in contact with a thermal bath (e.g., a radiation background) fixing the temperature, the statistical equilibrium state minimizes the free energy $F = E - TS$, or maximizes the Massieu function $J = S - \beta E$, at fixed mass and temperature:

$$\text{Max } J[f] \quad | \quad M[f] = M. \quad (15)$$

Introducing Lagrange multipliers and writing the variational principle in the form

$$\delta J + \frac{\mu}{T} \delta N = 0, \quad (16)$$

we find that the *critical points* of free energy are again given by the Fermi-Dirac distribution (14). Therefore, the critical points (first variations) of the variational problems (12) and (15) are the same. However, the stability of the system (regarding the second variations) can be different in microcanonical and canonical ensembles. When this happens, we speak of a situation of *ensemble inequivalence* [4]. The stability of the system can be determined by a graphical construction, by simply plotting the caloric curve/series of equilibria $\beta(E)$ and using the turning point method of Katz [22, 23].

2.2 Thermodynamical parameters

Integrating the distribution function (14) over velocity, we find that the density of particles is related to the gravitational potential by

$$\rho = \frac{\eta_0 S_D 2^{D/2-1}}{(\beta m)^{D/2}} I_{D/2-1}(\lambda e^{\beta m \Phi}), \quad (17)$$

where I_n denotes the Fermi integral

$$I_n(t) = \int_0^{+\infty} \frac{x^n}{1 + t e^x} dx. \quad (18)$$

We recall the identity

$$I'_n(t) = -\frac{n}{t} I_{n-1}(t), \quad (n > 0), \quad (19)$$

which can be established from (18) by an integration by parts. The gravitational potential is now obtained by substituting Eq. (17) in the Poisson equation (4). We introduce the rescaled

distance $\xi = [S_D^2 2^{D/2-1} G \eta_0 / (\beta m)^{D/2-1}]^{1/2} r$ and the variables $\psi = \beta m (\Phi - \Phi_0)$ and $k = \lambda e^{\beta m \Phi_0}$, where Φ_0 is the central potential. Thus, we get the D -dimensional Fermi-Poisson equation

$$\frac{1}{\xi^{D-1}} \frac{d}{d\xi} \left(\xi^{D-1} \frac{d\psi}{d\xi} \right) = I_{D/2-1}(k e^{\psi(\xi)}), \quad (20)$$

$$\psi(0) = \psi'(0) = 0. \quad (21)$$

As is well-known, self-gravitating systems at non-zero temperature have the tendency to evaporate. Therefore, there is no equilibrium state in a strict sense and the statistical mechanics of self-gravitating systems is essentially an out-of-equilibrium problem. However, the evaporation rate is small in general and the system can be found in a quasi-equilibrium state for a relatively long time. In order to describe the thermodynamics of the self-gravitating Fermi gas rigorously, we shall use an artifice and enclose the system within a spherical box of radius R (the box typically represents the size of the cluster under consideration). In that case, the solution of Eq. (20) is terminated by the box at the normalized radius

$$\alpha = \left[\frac{S_D^2 2^{D/2-1} G \eta_0}{(\beta m)^{D/2-1}} \right]^{1/2} R. \quad (22)$$

For a spherically symmetric configuration, the Gauss theorem can be written

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^{D-1}}, \quad (23)$$

where $M(r) = \int_0^r \rho S_D r'^{D-1} dr'$ is the mass within the sphere of radius r . Applying this result at $r = R$ and using the variables introduced previously we get

$$\eta \equiv \frac{\beta G M m}{R^{D-2}} = \alpha \psi'_k(\alpha). \quad (24)$$

This equation relates the dimensionless box radius α and the uniformizing variable k to the dimensionless inverse temperature η . According to Eqs. (22) and (24), α and k are related to each other by the relation $\alpha^2 \eta^{D/2-1} = \mu$ or, explicitly,

$$\alpha^{\frac{D+2}{D-2}} \psi'_k(\alpha) = \mu^{\frac{2}{D-2}}, \quad (25)$$

where

$$\mu = \eta_0 \sqrt{S_D^4 2^{D-2} G^D M^{D-2} R^{D(4-D)}}, \quad (26)$$

is the degeneracy parameter [20]. It should not be confused with the chemical potential. We shall give a physical interpretation of this parameter in Sec. 4.2.

The calculation of the energy is a little more involved. First, we introduce the local pressure

$$p = \frac{1}{D} \int f v^2 d^D \mathbf{v}. \quad (27)$$

Using the Fermi-Dirac distribution function (14), we find that

$$p = \frac{\eta_0 S_D 2^{D/2}}{D (\beta m)^{D/2+1}} I_{D/2}(k e^{\psi}). \quad (28)$$

The kinetic energy $K = (D/2) \int p d^D \mathbf{r}$ can thus be written

$$\frac{K R^{D-2}}{G M^2} = \frac{\alpha^{\frac{4+4D-D^2}{D-2}}}{\mu^{\frac{4}{D-2}}} \int_0^\alpha I_{D/2}(k e^{\psi_k(\xi)}) \xi^{D-1} d\xi. \quad (29)$$

In order to determine the potential energy, we use the D -dimensional version of the Virial theorem [12]. For $D \neq 2$, it reads

$$2K + (D - 2)W = DV_D R^D p(R), \quad (30)$$

where $V_D = S_D/D$ is the volume of a hypersphere with unit radius (the case $D = 2$ will be considered specifically in Sec. 4.6). Using the expression of the pressure (28) at the box radius R , we get

$$\frac{WR^{D-2}}{GM^2} = \frac{2}{D(D-2)} \frac{\alpha^{\frac{2(D+2)}{D-2}}}{\mu^{\frac{4}{D-2}}} I_{D/2}(ke^{\psi(\alpha)}) - \frac{2KR^{D-2}}{(D-2)GM^2}. \quad (31)$$

Combining Eqs. (29) and (31), we finally obtain

$$\Lambda \equiv -\frac{ER^{D-2}}{GM^2} = \frac{4-D}{D-2} \frac{\alpha^{\frac{4+4D-D^2}{D-2}}}{\mu^{\frac{4}{D-2}}} \int_0^\alpha I_{D/2}(ke^{\psi_k(\xi)}) \xi^{D-1} d\xi - \frac{2}{D(D-2)} \frac{\alpha^{\frac{2(D+2)}{D-2}}}{\mu^{\frac{4}{D-2}}} I_{D/2}(ke^{\psi(\alpha)}). \quad (32)$$

For $D = 3$, Eqs. (24) and (32) return the expressions derived in [20, 4]. For a given value of μ and k , we can solve the ordinary differential equation (20) until the value of α at which the condition (25) is satisfied. Then, Eqs. (24) and (32) determine the temperature and the energy of the configuration. By varying the parameter k (for a fixed value of the degeneracy parameter μ), we can determine the full caloric curve/series of equilibria $\beta(E)$. Extending the results of [4] in D dimensions, the entropy of each configuration, parameterized by α , is given by

$$\frac{S}{Nk_B} = -\frac{4+4D-D^2}{D(4-D)} \Lambda \eta + \psi_k(\alpha) + \frac{\eta}{D-2} + \ln k - \frac{2(D-2)}{D^2(4-D)} \frac{\alpha^{\frac{2D}{D-2}}}{\mu^{\frac{2}{D-2}}} I_{D/2}(ke^{\psi_k(\alpha)}), \quad (33)$$

and the free energy by

$$F = E - TS. \quad (34)$$

In the microcanonical ensemble, a solution is stable if it corresponds to a maximum of entropy $S[f]$ at fixed mass and energy. In the canonical ensemble, the condition of stability requires that the solution be a minimum of free energy $F[f]$ at fixed mass and temperature. This meanfield approach is *exact* in a thermodynamical limit such that $N \rightarrow +\infty$ with μ , η , Λ fixed. If we fix η_0 (i.e. \hbar) and G , this implies that $RN^{(D-2)/(D(4-D))}$, $TN^{-4/(D(4-D))}$, $EN^{-(4D-D^2+4)/(D(4-D))}$, SN^{-1} and JN^{-1} approach a constant value for $N \rightarrow +\infty$ (the free energy F scales as $N^{(4D-D^2+4)/(D(4-D))}$). This is the quantum thermodynamic limit (QTL) for the self-gravitating gas [4, 24]. The usual thermodynamic limit $N, R \rightarrow +\infty$ with N/R^D constant is clearly not relevant for inhomogeneous systems whose energy is non-additive.

3 Asymptotic limits

3.1 The non degenerate limit ($\mu = \infty$)

Before considering the case of an arbitrary degree of degeneracy, it may be useful to discuss first the non degenerate limit corresponding to a classical isothermal gas ($\hbar \rightarrow 0$). For $f \ll \eta_0$, the distribution function (14) reduces to the Maxwell-Boltzmann formula

$$f = \frac{\eta_0}{\lambda} e^{-\beta m(\frac{v^2}{2} + \Phi)}, \quad (35)$$

which can be written more conveniently as

$$f = \left(\frac{\beta m}{2\pi} \right)^{D/2} \rho(\mathbf{r}) e^{-\beta m \frac{v^2}{2}}. \quad (36)$$

The density profile can be written

$$\rho = \rho_0 e^{-\psi(\xi)}, \quad (37)$$

where ρ_0 is the central density, ξ is the normalized distance

$$\xi = (S_D G \beta m \rho_0)^{1/2} r, \quad (38)$$

and ψ is the solution of the D -dimensional Emden equation

$$\frac{1}{\xi^{D-1}} \frac{d}{d\xi} \left(\xi^{D-1} \frac{d\psi}{d\xi} \right) = e^{-\psi}, \quad (39)$$

with boundary conditions

$$\psi(0) = \psi'(0) = 0. \quad (40)$$

This equation can be obtained from Eq. (20) by taking the limit $k \rightarrow +\infty$ and using the limiting form of the Fermi integral

$$I_n(t) \sim \frac{1}{t} \Gamma(n+1), \quad (t \rightarrow +\infty). \quad (41)$$

From Eq. (36), we check that the local equation of state of a classical self-gravitating isothermal gas is $p(\mathbf{r}) = \frac{\rho(\mathbf{r})}{m} k_B T$ whatever the dimension of space. The thermodynamical parameters are given by

$$\eta = \alpha \psi'(\alpha), \quad (42)$$

$$\Lambda = \frac{D(4-D)}{2(D-2)} \frac{1}{\alpha \psi'(\alpha)} - \frac{1}{D-2} \frac{e^{-\psi(\alpha)}}{\psi'(\alpha)^2}, \quad (43)$$

$$\frac{S - S_0}{N k_B} = -\frac{D-2}{2} \ln \eta - 2 \ln \alpha + \psi(\alpha) + \frac{\eta}{D-2} - 2\Lambda\eta, \quad (44)$$

$$\frac{S_0}{N k_B} = \ln \mu + \ln \left(\frac{2\pi^{D/2}}{S_D} \right) + 1 - \frac{D}{2}, \quad (45)$$

where $\alpha = (S_D G \beta m \rho_0)^{1/2} R$ is the normalized box radius. For $D = 2$, the thermodynamical parameters can be calculated analytically [14]. Introducing the pressure at the box $P = p(R)$, the global equation of state of the self-gravitating gas can be written

$$\frac{PV}{N k_B T} = \frac{1}{D} \frac{\alpha^2}{\eta} e^{-\psi(\alpha)}. \quad (46)$$

We recall that the foregoing expressions can be expressed in terms of the value of the Milne variables $u_0 = u(\alpha)$ and $v_0 = v(\alpha)$ at the normalized box radius [7, 25]. The structure and the stability of classical isothermal spheres in D dimensions have been studied in detail in [14]. The classical thermodynamic limit (CTL) of self-gravitating systems, or dilute limit [17], is such that $N \rightarrow +\infty$ with η , Λ fixed. If we take $\beta \sim 1$, this implies that $R \sim N^{1/(D-2)}$ and $E, S, J, F \sim N$. The physical distinction between the quantum thermodynamic limit (QTL) and the classical thermodynamic limit (CTL) is related to the existence of long-lived gaseous metastable states as discussed in [24, 16].

3.2 The completely degenerate limit

For $\beta \rightarrow +\infty$ (i.e., $T = 0$), the distribution function (14) reduces to a step function: $f = \eta_0$ if $v \leq v_F$ and $f = 0$ if $v \geq v_F$, where $v_F(\mathbf{r}) = \sqrt{2(\mu/m - \Phi)}$ is the local Fermi velocity. In that case, the density and the pressure can be explicitly evaluated:

$$\rho = \int_0^{v_F} \eta_0 S_D v^{D-1} dv = \eta_0 S_D \frac{v_F^D}{D}, \quad (47)$$

$$p = \frac{1}{D} \int_0^{v_F} \eta_0 S_D v^{D+1} dv = \eta_0 \frac{S_D}{D} \frac{v_F^{D+2}}{D+2}. \quad (48)$$

Eliminating the Fermi velocity between these two expressions, we find that the equation of state of a cold Fermi gas in D dimensions is

$$p = K \rho^{1+2/D}, \quad K = \frac{1}{D+2} \left(\frac{D}{\eta_0 S_D} \right)^{2/D}. \quad (49)$$

This equation of state describes a D -dimensional classical white dwarf star (throughout this paper, we shall call “white dwarf star”, or “fermion ball”, a completely degenerate self-gravitating system. This terminology will be extended to any dimension of space). In $D = 3$, classical white dwarf stars are equivalent to polytropes with index $n = 3/2$ [11]. In D dimensions, classical “white dwarf stars” are equivalent to polytropes with index [12]:

$$n_{3/2} = \frac{D}{2}. \quad (50)$$

The structure and the stability of polytropic spheres in D dimensions have been studied in detail in [12]. It is shown that a polytrope of index n is self-confined for $n < n_5 = (D+2)/(D-2)$ and stable for $n < n_3 = D/(D-2)$. Therefore, white dwarf stars ($n = n_{3/2} = D/2$) are self-confined only for $D < 2(1 + \sqrt{2})$ and they are stable only for $D \leq 4$. For $D > 4$, quantum mechanics is not able to stabilize matter against gravitational collapse. Thus, $D = 4$ is a critical dimension regarding gravitational collapse. $D = 2$ is also critical [14]. Therefore, the dimension of space of our universe $2 < D = 3 < 4$ lies between two critical dimensions.

We now introduce dimensionless parameters associated with $n_{3/2}$ polytropes which will be useful in the sequel. Their density profile can be written

$$\rho(r) = \rho_0 \theta^{D/2}(\xi), \quad (51)$$

where ρ_0 is the central density, ξ is the normalized distance

$$\xi = \left[\frac{2S_D G \rho_0^{\frac{(D-2)}{D}}}{K(D+2)} \right]^{1/2} r, \quad (52)$$

and θ is solution of the D -dimensional Lane-Emden equation

$$\frac{1}{\xi^{D-1}} \frac{d}{d\xi} \left(\xi^{D-1} \frac{d\theta}{d\xi} \right) = -\theta^{D/2}, \quad (53)$$

with boundary conditions

$$\theta(0) = 1, \quad \theta'(0) = 0. \quad (54)$$

This equation can be obtained from Eq. (20) by taking the limit $k \rightarrow 0$ and using the limiting form of the Fermi integral

$$I_n(t) \sim \frac{(-\ln t)^{n+1}}{n+1}, \quad (t \rightarrow 0). \quad (55)$$

For $D < 2(1 + \sqrt{2})$, the solution of the Lane-Emden equation (53) vanishes at a finite distance ξ_1 defining the radius R_* of the white dwarf star (*complete polytrope*). Using the results of [12], the mass-radius relation of D -dimensional white dwarf stars is given by

$$M^{\frac{D-2}{D}} R_*^{4-D} = \frac{K(D+2)}{2GS_D^{2/D}} \omega_{D/2}^{\frac{D-2}{D}}, \quad (56)$$

where we have defined

$$\omega_{D/2} = -\xi_1^{\frac{D+2}{D-2}} \theta'(\xi_1). \quad (57)$$

For $2 < D < 4$, the mass M decreases with the radius R_* while for $D < 2$ and for $4 < D < 2(1 + \sqrt{2})$ it increases with the radius (see Fig. 1). The mass-radius relation (56) exhibits the two critical dimensions of space $D = 2$ and $D = 4$ discussed previously. For $D = 2$, the radius is independent on mass and for $D = 4$, the mass is independent on radius (see Sec. 4). The energy of a self-confined white dwarf star is

$$E = -\lambda_{D/2} \frac{GM^2}{R_*^{D-2}}, \quad (58)$$

where

$$\lambda_{D/2} = \frac{D(4-D)}{(D-2)(4+4D-D^2)}. \quad (59)$$

We note that the energy of a white dwarf star vanishes for $D = 4$. According to Poincaré's theorem [10], this determines the onset of instability. We thus recover the fact that complete white dwarf stars are unstable for $D > 4$ [12].

For $D > 2(1 + \sqrt{2})$, the density of a $n_{3/2}$ polytrope never vanishes (as $n_{3/2} > n_5$) and we need to confine the system within a box of radius R (*incomplete polytrope*) to avoid the infinite mass problem. In that case, the white dwarf star exerts a pressure against the box. White dwarf stars with $R_* > R$ when $D < 2(1 + \sqrt{2})$ are also incomplete. They are arrested by the box at the normalized radius $\xi = \alpha$ with $\alpha = \{2S_D G \rho_0^{(D-2)/D} / [K(D+2)]\}^{1/2} R$. As shown in [12], the normalized mass and the normalized energy of the configuration parameterized by α are given by

$$\eta_P \equiv \frac{M}{S_D} \left[\frac{2S_D G}{K(D+2)} \right]^{\frac{D}{D-2}} \frac{1}{R^{\frac{D(D-4)}{D-2}}} = -\alpha^{\frac{D+2}{D-2}} \theta'(\alpha), \quad (60)$$

$$\Lambda \equiv -\frac{ER^{D-2}}{GM^2} = \frac{-2}{D^2 - 4D - 4} \left\{ \frac{D(4-D)}{2(D-2)} \left[1 + (D-2) \frac{\theta(\alpha)}{\alpha \theta'(\alpha)} \right] + \frac{2-D}{2+D} \frac{\theta(\alpha)^{\frac{D+2}{2}}}{\theta'(\alpha)^2} \right\}. \quad (61)$$

In the present context, the normalized mass η_P is related to the degeneracy parameter μ by the relation

$$\eta_P = \left(\frac{2\mu}{D} \right)^{\frac{2}{D-2}}. \quad (62)$$

On the other hand, using Eqs. (56) and (58), the normalized mass and the normalized energy of a self-confined white dwarf star with $R_* < R$ (complete polytrope) are given by

$$\eta_P = \omega_{D/2} \left(\frac{R_*}{R} \right)^{\frac{(D-4)D}{D-2}} \quad (63)$$

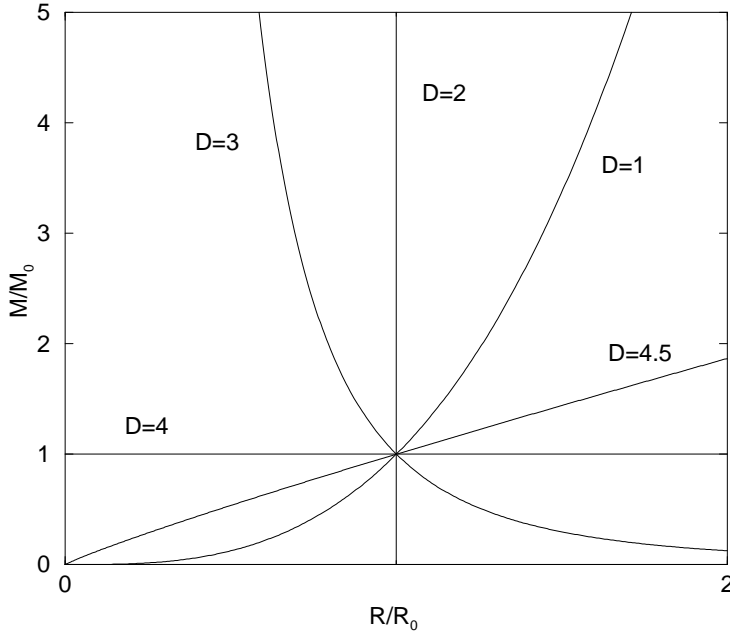


Figure 1: The mass-radius relation for complete white dwarf stars ($T = 0$) in different dimensions of space. It clearly shows that the dimension $D = 3$ is surrounded by two critical dimensions $D = 2$ and $D = 4$ at which either the radius or the mass is constant.

$$\Lambda = \lambda_{D/2} \left(\frac{R}{R_*} \right)^{D-2}. \quad (64)$$

Eliminating R_* between these two relations, we obtain the “mass-energy” relation

$$\Lambda \eta_P^{\frac{(D-2)^2}{D(D-4)}} = \lambda_{D/2} (\omega_{D/2})^{\frac{(D-2)^2}{D(D-4)}}, \quad (65)$$

which will be useful in our subsequent analysis.

4 Caloric curves in various dimensions

4.1 Series of equilibria and metastable states

We shall now determine the caloric curve $\beta(E)$ of the self-gravitating Fermi gas as a function of the degeneracy parameter μ for any dimension of space D . This study has already been performed for $D = 3$ in [4]. The critical points of the Fermi-Dirac entropy $S[f]$ at fixed E and M (i.e., the distribution functions $f(\mathbf{r}, \mathbf{v})$ which cancel the first order variations of S at fixed E, M) form a series of equilibria parameterized by the uniformizing variable k . At each point in the series of equilibria corresponds a temperature β and an energy E determined by Eqs. (24) and (32). In this approach, β is the Lagrange multiplier associated with the conservation of energy in the variational problem (13). It has also the interpretation of a kinetic temperature in the Fermi-Dirac distribution (14). We can thus plot $\beta(E)$ along the series of equilibria. There can be several values of temperature β for the same energy E because the variational problem (12) can have several solutions: a local entropy maximum (metastable state), a global entropy maximum, and one or several saddle points. We shall represent all these solutions on the caloric curve because local entropy maxima (metastable states) are in general more physical than global entropy maxima for the timescales achieved in astrophysics. Indeed, the system

can remain frozen in a metastable gaseous phase for a very long time. This is the case, in particular, for globular clusters and for the gaseous phase of fermionic matter (at high energy and high temperature). The time required for a metastable gaseous system to collapse is in general tremendously long and increases exponentially with the number N of particles (thus, $t_{life} \rightarrow +\infty$ in the thermodynamic limit $N \rightarrow +\infty$) [16]. This is due to the long-range nature of the gravitational potential. Therefore, metastable states are in reality stable states. At high temperatures and high energies, the global entropy maximum is not physically relevant [26, 27, 25, 24]. Condensed objects (e.g., planets, stars, white dwarfs, fermion balls,...) only form below a critical energy E_c (Antonov energy) [8, 9, 2] or below a critical temperature T_c (Jeans temperature) [7], when the gaseous metastable phase ceases to exist (spinodal point).

4.2 The case $2 < D < 4$

We start to describe the structure of the caloric curve of the self-gravitating Fermi gas for $2 < D < 4$ (specifically $D = 3$). Let us first consider the Fermi gas at $T = 0$ (white dwarf stars). The $\Lambda - \eta_P$ curve defined by Eqs. (60), (61) and (65) is represented in Fig. 2. In the present context, it gives the energy of the star as a function of its mass. Since the curve does not present turning points, all the white dwarf star configurations are stable. According to Eq. (56), for $2 < D < 4$, the mass M of a complete white dwarf star is a decreasing function of its radius R_* . Therefore, if the system is enclosed within a box, there exists a characteristic mass

$$M_*(R) = \frac{\chi D}{\eta_0^{\frac{2}{D-2}} G^{\frac{D}{D-2}}} R^{-\frac{D(4-D)}{D-2}} \quad (66)$$

such that for $M > M_*(R)$ the star is self-confined ($R_* < R$) and for $M < M_*(R)$, it is restricted by the box. In terms of the dimensionless mass η_P , complete $n_{3/2}$ polytropes correspond to $\eta_P \geq \omega_{D/2}$ and incomplete $n_{3/2}$ polytropes to $\eta_P \leq \omega_{D/2}$. For $2 < D < 4$, there exists a stable equilibrium at $T = 0$ for all mass M .

We now briefly describe the caloric curve for arbitrary temperature and energy. A more complete description is given in [4] for $D = 3$. First, we note that, according to Eqs. (26), (49) and (56),

$$\mu = \mu_*(D) \left(\frac{R}{R_*} \right)^{\frac{D(4-D)}{2}}, \quad (67)$$

where

$$\mu_*(D) \equiv \frac{D}{2} (\omega_{D/2})^{\frac{D-2}{2}}. \quad (68)$$

Therefore, the degeneracy parameter μ can be seen as the ratio (with some power) between the size of the system R and the size R_* of a white dwarf star with mass M . Accordingly, a small value of μ corresponds to a large “effective” cut-off (played by Pauli’s exclusion principle) or, equivalently, to a small system size. Alternatively, a large value of μ corresponds to a small “effective” cut-off or a large system size. This gives a physical interpretation to the degeneracy parameter. For $\mu \rightarrow +\infty$ (i.e. $\hbar \rightarrow 0$), we recover classical isothermal spheres. In that case, the caloric curve $\beta(E)$ forms a spiral. For finite values of μ , the spiral unwinds due to the influence of degeneracy and gives rise to a rich variety of caloric curves (Fig. 3). For large systems, the caloric curve has a Z -shape (“dinosaur’s neck”) and for small systems it has a N -shape. The phase transitions in the self-gravitating Fermi gas for $D = 3$ and the notion of metastable states, spinodal points, critical points, collapse, explosion, and hysteresis are discussed in [4, 27, 24, 16]. Similar notions are discussed in [28] for a hard spheres gas. The ground state of the self-gravitating Fermi gas ($T = 0$) corresponds to a white dwarf star

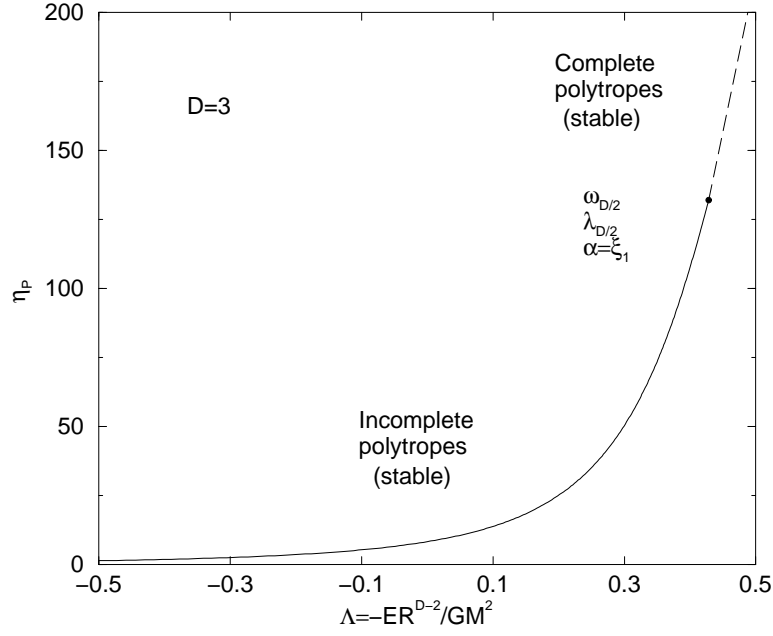


Figure 2: The mass-energy relation for white dwarf stars ($T = 0$) in $D = 3$. There exists an equilibrium state for all mass. The white dwarf star is self-confined if $M > M_*(R)$ and box-confined if $M < M_*(R)$.

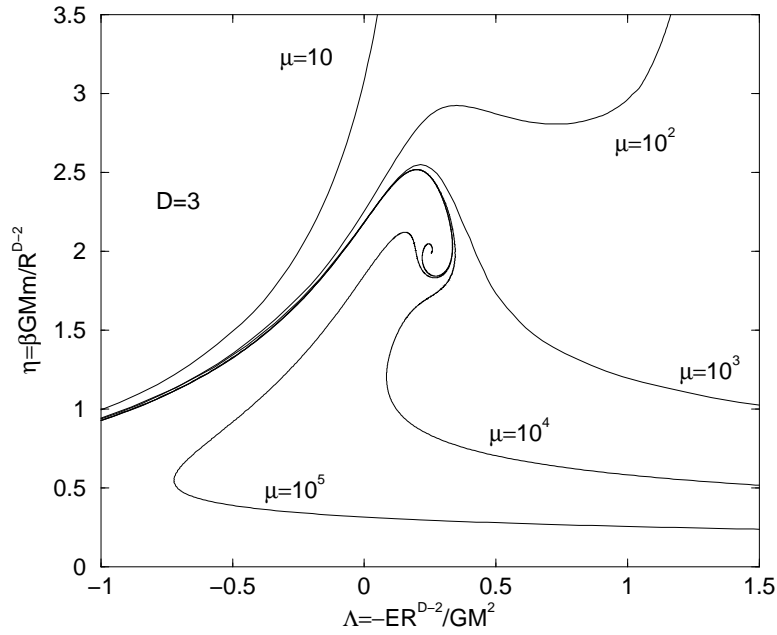


Figure 3: Caloric curve in $D = 3$ for different values of the degeneracy parameter (various system sizes).

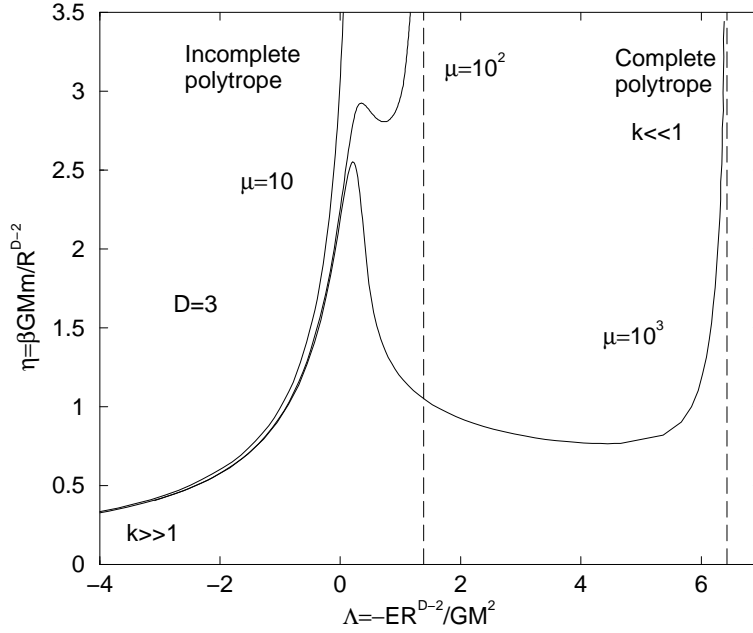


Figure 4: Caloric curve in $D = 3$ for small values of the degeneracy parameter (small system sizes). For $D < 4$, there exists an equilibrium state for all temperatures T and all accessible energies $E \geq E_{ground}$.

configuration. For given μ , its structure (radius, energy) is determined by the intersection between the $\Lambda - \eta_P$ curve in Fig. 2 and the line defined by Eq. (62). The “white dwarf” is complete ($R_* < R$) for $\mu > \mu_*(D)$ and incomplete ($R_* > R$) otherwise. For $\mu > \mu_*(D)$, the normalized energy of the white dwarf is given by

$$\Lambda_{max}(D, \mu) = \lambda_{D/2} \left(\frac{\mu}{\mu_*} \right)^{\frac{2(D-2)}{D(4-D)}}. \quad (69)$$

This is the ground state of the self-gravitating Fermi gas corresponding to the asymptote in Fig. 4 (this asymptote exists for all curves in Fig. 3 but is outside the frame). For classical particles ($\hbar = 0$), there is no equilibrium state if energy and temperature are below a critical threshold [8, 9]. In that case, the system undergoes gravitational collapse and forms binaries (in microcanonical ensemble) or a Dirac peak (in canonical ensemble); see Appendices A and B of [14] and [25, 24, 16]. For self-gravitating fermions, an equilibrium state exists for all values of temperature and for all accessible energies ($E \geq E_{ground}$). Gravitational collapse is arrested by quantum pressure as first realized by Fowler [11]. We shall now show that this claim ceases to be true in dimension $D \geq 4$.

4.3 The case $4 < D < 2(1 + \sqrt{2})$

We now consider the case $4 < D < 2(1 + \sqrt{2})$ (specifically $D = 4.1$). Let us first describe the Fermi gas at $T = 0$. The $\Lambda - \eta_P$ curve defined by Eqs. (60), (61) and (65) is represented in Fig. 5. For $D > 4$, the curves $\eta_P(\alpha)$ and $\Lambda(\alpha)$ associated to $n_{3/2}$ polytropes have their extrema at the same point (see Appendix C of [12]). Therefore, the $\Lambda - \eta_P$ curve presents a cusp at $(\Lambda_0, \eta_{P,c})$. Past this point in the series of equilibria, $n_{3/2}$ polytropes are unstable. According to Eq. (56), for $D > 4$, the radius R_* of a self-confined white dwarf star increases with its mass. For $M < M_*(R)$ there exists self-confined white dwarf star configurations. In

terms of the dimensionless mass η_P , this corresponds to $\eta_P \leq \omega_{D/2}$ (see Fig. 5). However, such configurations are unstable since they lie after the turning point [12]. Therefore, only incomplete (box confined) white dwarf stars can be stable in $D > 4$. Inspecting Fig. 5 again, we observe that these configurations exist only below a critical mass

$$M_c(R) = \eta_{P,c}(D) S_D R^{\frac{D(D-4)}{D-2}} \left[\frac{K(D+2)}{2S_D G} \right]^{\frac{D}{D-2}}. \quad (70)$$

For $M > M_c(R)$, there is no equilibrium state at $T = 0$ for $D > 4$. In terms of the dimensionless mass η_P , equilibrium states exist only for $\eta_P < \eta_{P,c}(D)$.

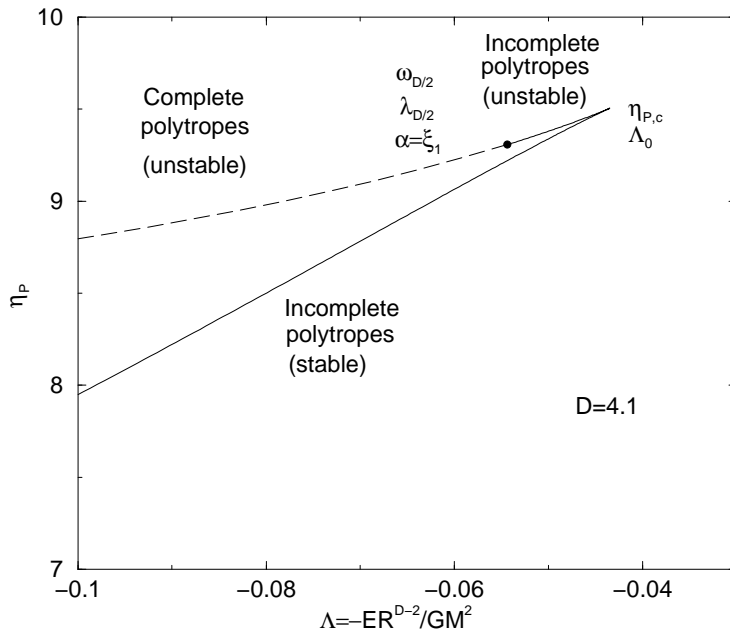


Figure 5: The mass-energy relation for white dwarf stars ($T = 0$) in $4 < D < 2(1 + \sqrt{2})$ (specifically $D = 4.1$). Self-confined white dwarf stars are always unstable. Box-confined white dwarf stars exist only for $M < M_c(R)$. For $M > M_c(R)$, there is no equilibrium state.

The caloric curve for arbitrary value of temperature and energy is represented in Fig. 6. For $\mu \rightarrow +\infty$, we recover the classical spiral [14]. For finite values of μ , there exists equilibrium solutions at all temperatures only if $\eta_P < \eta_{P,c}(D)$. Using Eq. (62), this corresponds to

$$\mu < \frac{D}{2} \eta_{P,c}(D)^{\frac{D-2}{2}} \equiv \mu_c(D). \quad (71)$$

If $\mu > \mu_c(D)$, or equivalently if $M > M_*(R)$, there exists a minimum energy $E_c = -\Lambda_c GM^2/R^{D-2}$ (which appears to be positive) and a minimum temperature $T_c = GM/(\eta_c R^{D-2})$ below which there is no equilibrium state (the values of η_c and Λ_c depend on D and μ). In that case, the system is expected to collapse. This is similar to the Antonov instability (gravothermal catastrophe) for classical particles [8, 9]. Since we deal here with self-gravitating fermions, we could expect that quantum pressure would arrest the collapse. Our study shows that this is not the case for $D > 4$. Quantum mechanics cannot stabilize matter against gravitational collapse anymore.

4.4 The case $D = 4$

The dimension $D = 4$ is special because it is the dimension of space above which quantum pressure cannot balance gravity anymore. Therefore, $D = 4$ is critical and it deserves a partic-

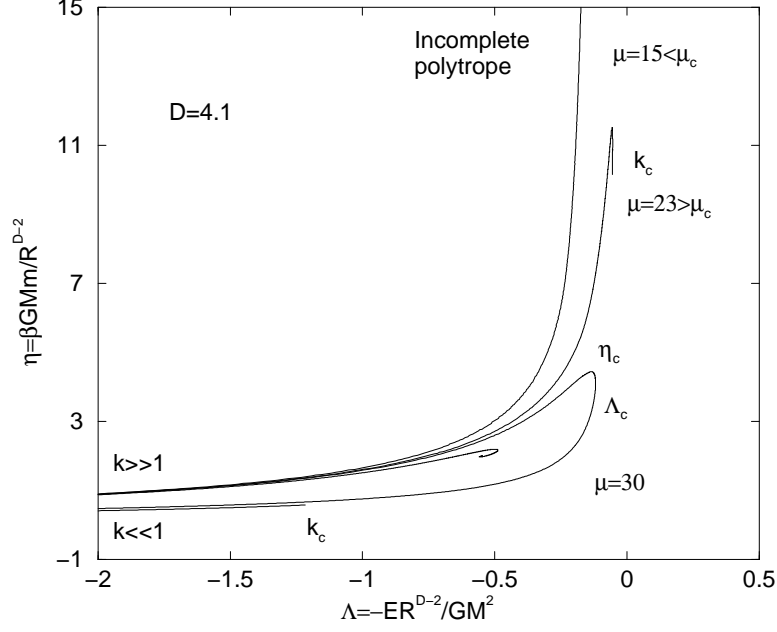


Figure 6: Caloric curve in $D = 4.1$ for different values of the degeneracy parameter. For $\mu > \mu_c(D)$, there is no equilibrium state if the temperature and the energy are too low. The reason for the “gap” at k_c is explained in Fig. 7.

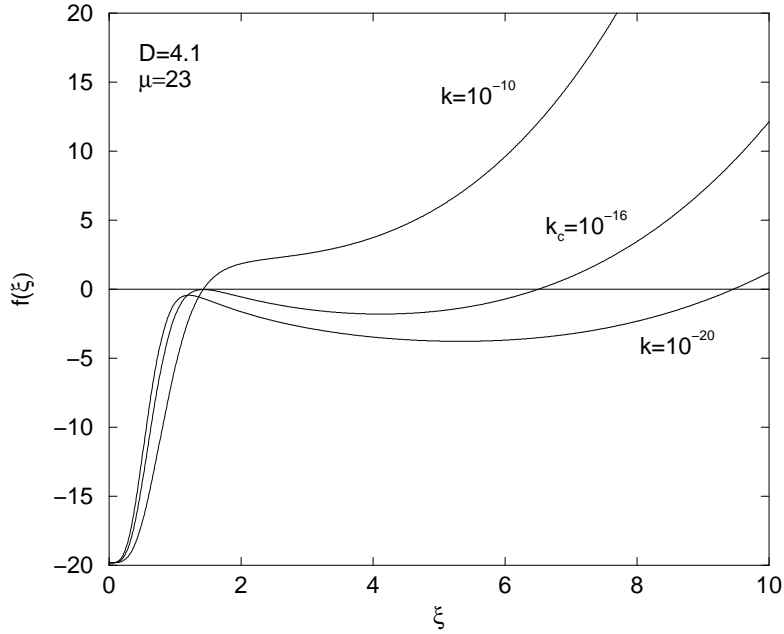


Figure 7: Graphical construction determining the value of α for given μ and k (in $D = 4.1$). According to Eq. (25), the normalized box radius α is solution of $f(\alpha) = 0$, where $f(\xi) = \xi^{\frac{D+2}{D-2}} \psi'_k(\xi) - \mu^{\frac{2}{D-2}}$. We see that α undergoes a discontinuity as $k \rightarrow k_c$. This gives rise to the “gap” in Fig. 6 for $\mu = 23$. However, this gap is essentially a mathematical curiosity since the lower part of the curve (small k) is unstable anyway.

ular attention. First, consider the Fermi gas at $T = 0$. It corresponds to a polytrope of index $n_{3/2} = n_3$ [12]. The $\Lambda - \eta_P$ curve defined by Eqs. (60), (61) and (65) is represented in Fig. 8. Since the curve is monotonic the box-confined $n_{3/2}$ polytropes are stable and the complete $n_{3/2}$ polytropes are marginally stable. For $D = 4$, the mass of a self-confined white dwarf star is independent on its radius, see Eq. (56). It can be expressed in terms of fundamental constants as

$$M_{limit} = \frac{\omega_2}{g S_4^2} \frac{h^4}{m^5 G^2} \simeq 1.44 \cdot 10^{-2} \frac{h^4}{m^5 G^2}, \quad (72)$$

where $\omega_2 \simeq 11.2$ (we have taken $g = 2$ in the numerical application). Mathematically, this is similar to Chandrasekhar's limiting mass for relativistic white dwarf stars equivalent to $n = 3$ polytropes in $D = 3$ [15]. However, it is here a purely classical (i.e. nonrelativistic) result. Relativistic effects will be considered in a forthcoming paper [29]. The energy of the self-confined white dwarf stars is $E = 0$. Considering Fig. 8 again, we see that incomplete white dwarf stars exist only for $M < M_{limit}$. In terms of the dimensionless mass η_P , this corresponds to $\eta < \eta_{P,c} = \omega_2 \simeq 11.2$. For $M > M_{limit}$, there is no equilibrium state at $T = 0$. The caloric curve for arbitrary value of temperature and energy is represented in Fig. 9 (see an enlargement in Fig. 10). Its description is similar to that of Sec 4.3. For $M > M_{limit}$, or equivalently $\mu \geq \mu_c = 2\omega_2 \simeq 22.4$, there exists a minimum energy $E_c = -\Lambda_c GM^2/R^2$ and a minimum temperature $T_c = GM/(\eta_c R^2)$ below which there is no equilibrium state.

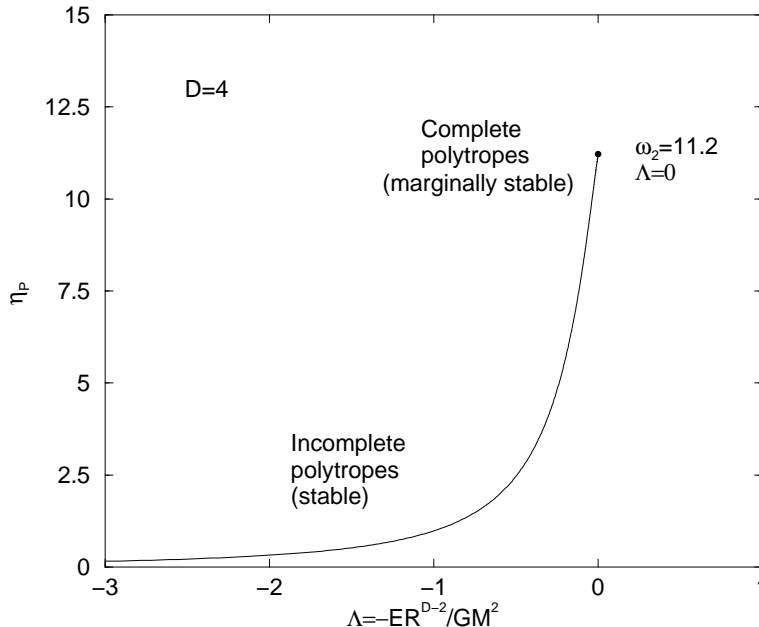


Figure 8: The mass-energy relation for white dwarf stars ($T = 0$) in $D = 4$. Self-confined white dwarf stars are marginally stable. They have a unique mass M_{limit} independent on their radius. For $M < M_{limit}$, the white dwarf star is box-confined. There is no equilibrium state with $M > M_{limit}$.

4.5 The case $D \geq 2(1 + \sqrt{2})$

The caloric curves for $D \geq 2(1 + \sqrt{2})$ are similar to those of Secs. 4.3 and 4.4. There are, however, two main differences. For $D \geq 10$, the classical spiral ceases to exist [14]. Thus, the caloric curve does not wind up as $\mu \rightarrow +\infty$ contrary to Fig. 10. On the other hand, for $D \geq 2(1 + \sqrt{2})$, it is not possible to construct self-confined white dwarf stars [12]. This is just a

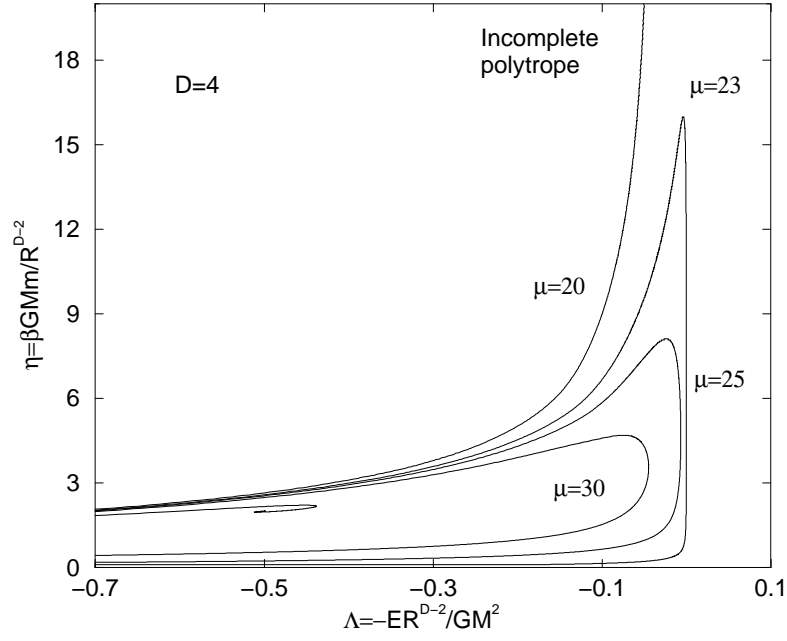


Figure 9: Caloric curves in $D = 4$ for different values of the degeneracy parameter. For $\mu > \mu_c = 22.4$, there is no equilibrium state if the temperature and the energy are too low.

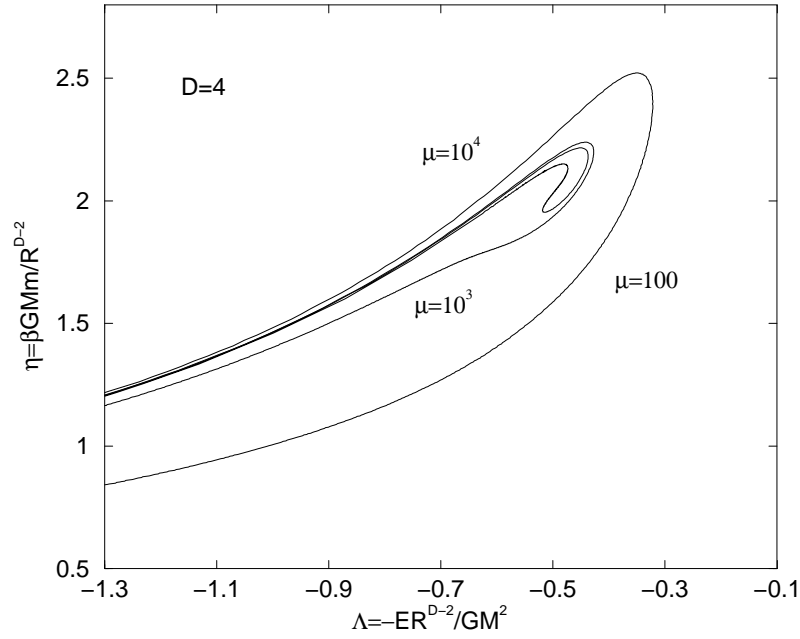


Figure 10: Same as Fig. 9 for larger values of μ showing the development of the classical spiral recovered for $\mu \rightarrow +\infty$.

mathematical curiosity since complete white dwarf stars are unstable for $D > 4$ anyway. This property changes the unstable branch of the $\Lambda - \eta_P$ diagram without consequence on the caloric curves. The $\Lambda - \eta_P$ diagram is represented Figs. 11 and 12. For $D > 2(1 + \sqrt{2})$, it displays an infinity of cusps towards the singular solution $(\Lambda_s, \eta_{P,s})$, see Fig. 12. For $D = 2(1 + \sqrt{2})$, there is just one cusp (see Fig. 11) and the Lane-Emden equation (53) can be solved analytically. This corresponds to the D -dimensional Schuster solution obtained for $n = n_5$ [12]. In that case, we find explicitly

$$\theta_5 = \frac{1}{\left[1 + \frac{\xi^2}{4(2+\sqrt{2})}\right]^{\sqrt{2}}}. \quad (73)$$

The normalized mass and the normalized energy can be expressed as

$$\eta_P = \frac{\alpha^{2+\sqrt{2}}}{2(1 + \sqrt{2}) \left[1 + \frac{\alpha^2}{4(2+\sqrt{2})}\right]^{1+\sqrt{2}}}, \quad (74)$$

$$\Lambda_5 = -2(1 + \sqrt{2}) \left[1 + \frac{\alpha^2}{4(2 + \sqrt{2})}\right]^{2(1+\sqrt{2})} \frac{1}{\alpha^{2(2+\sqrt{2})}} \int_0^\alpha \frac{\xi^{1+2\sqrt{2}}}{\left[1 + \frac{\xi^2}{4(2+\sqrt{2})}\right]^{2(1+\sqrt{2})}} d\xi. \quad (75)$$

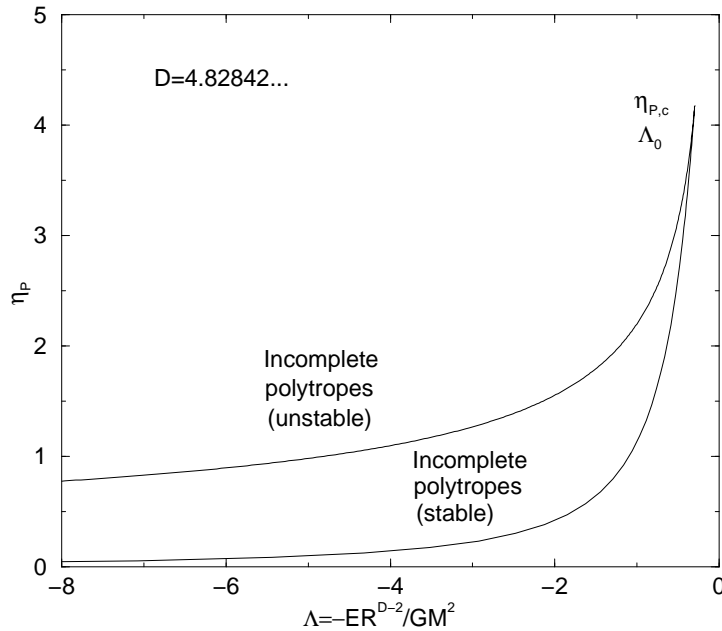


Figure 11: The mass-energy relation for white dwarf stars ($T = 0$) in $D = 2(1 + \sqrt{2})$.

4.6 The case $D = 2$

Let us now consider smaller dimensions of space. The dimension $D = 2$ is critical concerning gravitational collapse as discussed in [14]. For $D = 2$, the relevant Fermi integrals are I_0 and I_1 . By definition,

$$I_0(t) = \int_0^{+\infty} \frac{dx}{1 + te^x}. \quad (76)$$

Changing variables to $y = e^x$, we easily find that

$$I_0(t) = \ln\left(1 + \frac{1}{t}\right). \quad (77)$$

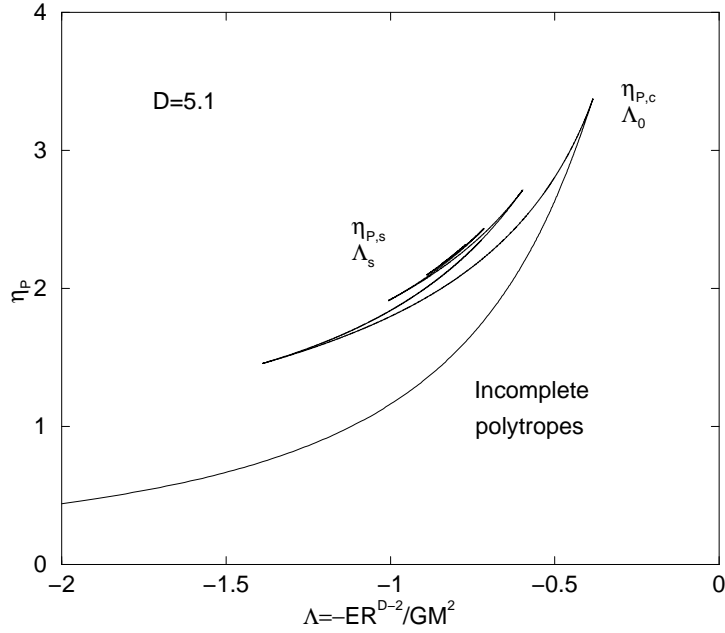


Figure 12: The mass-energy relation for white dwarf stars ($T = 0$) in $D = 5.1$

Therefore, the Fermi-Poisson equation (20) becomes

$$\frac{1}{\xi} \frac{d}{d\xi} \left(\xi \frac{d\psi}{d\xi} \right) = \ln(1 + k^{-1} e^{-\psi}). \quad (78)$$

$$\psi(0) = \psi'(0) = 0. \quad (79)$$

On the other hand, using the identity (19), giving

$$I_1'(t) = -\frac{1}{t} \ln\left(1 + \frac{1}{t}\right), \quad (80)$$

one finds that

$$I_1(t) = -\int_{-1/t}^0 \frac{\ln(1-x)}{x} dx = -\text{Li}_2\left(-\frac{1}{t}\right), \quad (81)$$

where Li_2 is the dilogarithm.

Consider first the Fermi gas at $T = 0$. In $D = 2$, a white dwarf star is equivalent to a polytrope with index $n_{3/2} = 1$. The Lane-Emden equation can then be solved analytically and we obtain $\theta = J_0(\xi)$, where J_0 is the Bessel function of zeroth order. The density drops to zero at $\xi_1 = \alpha_{0,1} \simeq 2.40$, the first zero of J_0 . Considering the mass-radius relation (56) in $D = 2$, we see that the radius is independant on mass. Therefore, complete white dwarf stars in two dimensions all have the same radius. It can be written in terms of fundamental constants as

$$R_* = \frac{\xi_1}{2\pi} \left(\frac{h^2}{gm^3G} \right)^{1/2} = 0.27 \frac{h}{m^{3/2}G^{1/2}}. \quad (82)$$

The relation between the mass and the central density of the white dwarf star is

$$M = \frac{\rho_0}{4\pi^2} \frac{h^2}{gm^3G} \xi_1 |\theta_1'|, \quad (83)$$

where $\theta'_1 = J'_0(\alpha_{0,1}) \simeq -0.52$. Thus, the density profile of a two-dimensional white dwarf star can be written

$$\rho(r) = \rho_0 J_0\left(\frac{\xi_1 r}{R_*}\right). \quad (84)$$

This is similar to the vorticity profile of a minimum enstrophy vortex in 2D hydrodynamics [30, 31]. The energy of a complete polytrope of index n in $D = 2$ is $E = -(n-1)GM^2/8 + (1/2)GM^2 \ln(R_*/R)$ with the convention $\Phi(R) = 0$ [12]. Therefore the energy of a 2D white dwarf star is

$$E = \frac{1}{2}GM^2 \ln\left(\frac{R_*}{R}\right). \quad (85)$$

Two-dimensional white dwarf stars exist for any mass M and they are stable. Noting that $R_*/R = (\mu_*/\mu)^{1/2} = \xi_1/\sqrt{\mu}$ where $\mu = 4\pi^2\eta_0 GR^2$, we can write the normalized energy of the self-confined white dwarf star as

$$\Lambda = \frac{1}{2} \ln\left(\frac{\sqrt{\mu}}{\xi_1}\right). \quad (86)$$

Let us now consider the case of incomplete white dwarf stars that are confined by the box ($R_* > R$). This corresponds to $\mu < \xi_1^2$. Using Eq. (52), we find that $\alpha = \sqrt{\mu}$. Then, using the results of [12], we find that the normalized energy of a box-confined white dwarf star in two dimensions is

$$\Lambda = -\frac{1}{2} \frac{J_0(\sqrt{\mu})}{\sqrt{\mu} J_1(\sqrt{\mu})}. \quad (87)$$

We now consider the self-gravitating Fermi gas at finite temperature $T \neq 0$. According to Eq. (22) we have $\alpha = \sqrt{\mu}$. Using Eq. (24), we obtain

$$\eta \equiv \beta GMm = \sqrt{\mu} \psi'(\sqrt{\mu}). \quad (88)$$

We need to calculate the energy specifically because the expression (32) breaks down in $D = 2$. The kinetic energy $K = \int p d^2\mathbf{r}$ can be written

$$\frac{K}{GM^2} = \frac{1}{\eta^2} \int_0^{\sqrt{\mu}} I_1(ke^\psi) \xi d\xi. \quad (89)$$

On the other hand, using an integration by parts, the potential energy is given by

$$W = -\frac{1}{4\pi G} \int (\nabla\Phi)^2 d^2\mathbf{r}, \quad (90)$$

where we have taken $\Phi(R) = 0$. Introducing the dimensionless quantities defined in Sec. 2.2, we get

$$\frac{W}{GM^2} = -\frac{1}{2\eta^2} \int_0^{\sqrt{\mu}} \psi'(\xi)^2 \xi d\xi. \quad (91)$$

Summing Eqs. (89) and (91), the total normalized energy of the Fermi gas in two dimensions is

$$\Lambda \equiv -\frac{E}{GM^2} = -\frac{1}{\eta^2} \int_0^{\sqrt{\mu}} I_1(ke^\psi) \xi d\xi + \frac{1}{2\eta^2} \int_0^{\sqrt{\mu}} \psi'(\xi)^2 \xi d\xi. \quad (92)$$

The corresponding caloric curve is plotted in Fig. 13. For $\mu \rightarrow +\infty$, we recover the classical caloric curve displaying a critical temperature $k_B T_c = GMm/4$ [14]. Below T_c , a classical gas experiences a gravitational collapse and develops a Dirac peak [14]. When quantum mechanics is taken into account, the collapse stops when the system becomes degenerate. The Dirac peak is replaced by a fermion ball surrounded by a dilute halo. At $T = 0$, we have a pure

Fermi condensate without halo. This is the ground state of the self-gravitating Fermi gas corresponding to the vertical asymptotes in Fig. 13. For $\mu < \xi_1^2$ (incomplete white dwarf stars), the minimum energy is given by Eq. (87) and for $\mu < \xi_1^2$ (complete white dwarf stars) by Eq. (86). This discussion concerning the difference between Dirac peaks (for classical particles) and fermion balls (for quantum particles) in the canonical ensemble remains valid for $2 \leq D < 4$. Note also that there is no collapse (gravothermal catastrophe) in the microcanonical ensemble in $D = 2$ [32, 14].

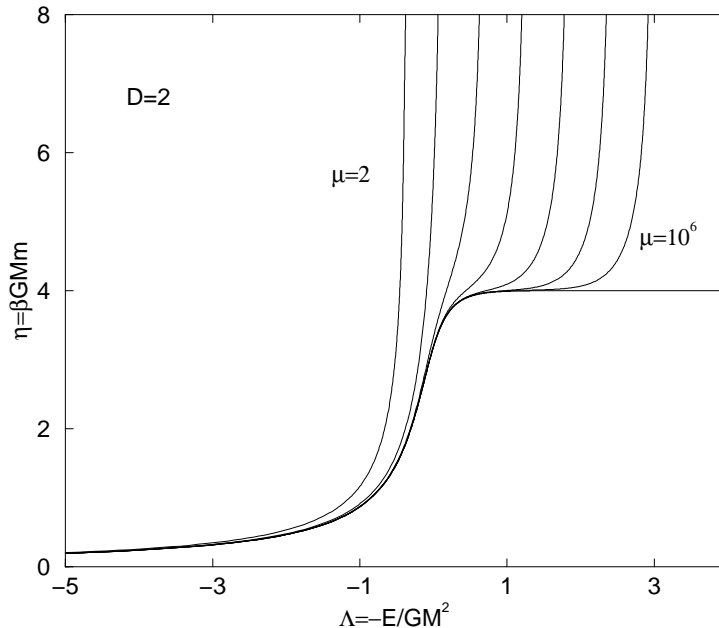


Figure 13: Caloric curve in $D = 2$ for different values of the degeneracy parameter: $\mu = 2, 10, 100, 10^3, 10^4, 10^5, 10^6$. For $\mu \rightarrow +\infty$, we recover the classical caloric curve displaying a critical temperature T_c . Below T_c the system is expected to collapse and create a Dirac peak (“black hole”). When quantum mechanics is accounted for, the “black hole” is replaced by a “fermion ball”. This result is generally valid for $2 \leq D < 4$.

4.7 The case $D < 2$

We finally conclude by the case $D < 2$ (specifically $D = 1$). First, we consider the Fermi gas at $T = 0$. The $\Lambda - \eta_P$ curve which gives the energy of the star as a function of its mass is represented in Fig. 14. Since the curve does not present turning points, all the white dwarf star configurations are stable. According to Eq. (56), for $D < 2$, the mass M of a complete white dwarf star increases with its radius R_* . Therefore, for $M < M_*(R)$ the star is self-confined and for $M > M_*(R)$ it is restricted by the box. There exists a stable equilibrium state at $T = 0$ for all mass. In terms of the dimensionless mass η_P , complete $n_{3/2}$ polytropes correspond to $\eta_P \leq \omega_{D/2}$ and incomplete $n_{3/2}$ polytropes to $\eta_P \geq \omega_{D/2}$. This situation is reversed with respect to that of Fig. 2.

The caloric curve for arbitrary temperature and energy is represented in Fig. 15. For $\mu \rightarrow +\infty$ (i.e. $\hbar \rightarrow 0$), we recover the curve obtained in [14] for classical isothermal systems. The caloric curve $\beta(E)$ is monotonic. Therefore, there is no phase transition for $D < 2$. Thus, the change in the caloric curve due to quantum mechanics is not very important since an equilibrium state (global maximum of entropy or free energy) already exists for any accessible energy E and any temperature T in classical mechanics. Quantum mechanics, however, changes

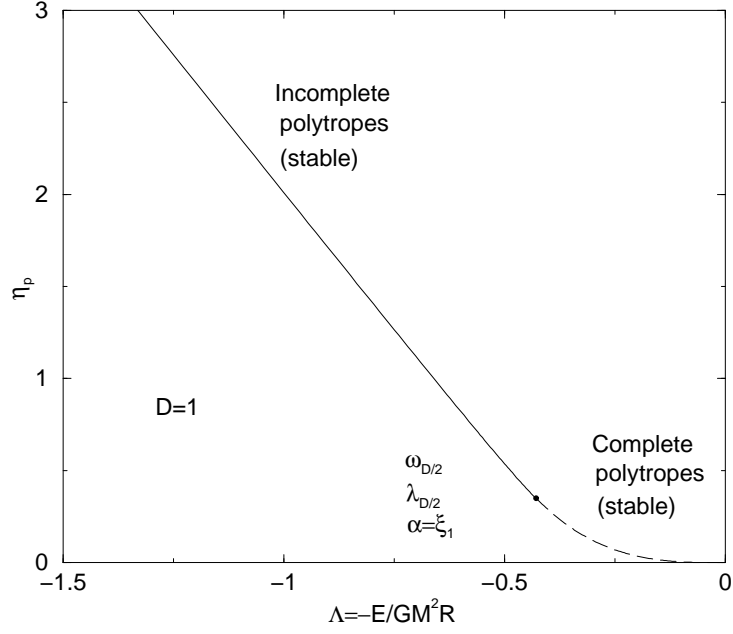


Figure 14: The mass-energy relation for white dwarf stars ($T = 0$) in $D < 2$ (specifically $D = 1$). There exists an equilibrium state for all mass. The white dwarf star is self-confined if $M < M_*(R)$ and box-confined if $M > M_*(R)$.

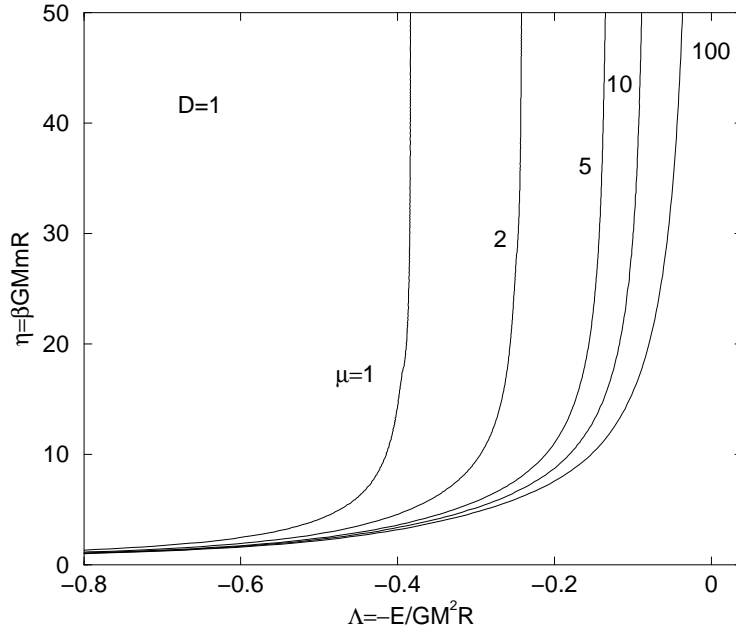


Figure 15: Caloric curve in $D = 1$ for different values of the degeneracy parameter (various system sizes).

the ground state of the system. The ground state of the self-gravitating Fermi gas ($T = 0$) corresponds to a white dwarf star configuration. Its structure (radius, energy) is determined by the intersection between the $\Lambda - \eta_P$ curve in Fig. 14 and the line defined by Eq. (62). The “white dwarf” is complete ($R_* < R$) for $\mu > \mu_*(D)$ and incomplete ($R_* > R$) otherwise. For $\mu > \mu_*(D)$, the normalized energy of the white dwarf is given by Eq. (69). This is the ground state of the self-gravitating Fermi gas corresponding to the asymptote in Fig. 15. In $D = 1$, it is possible to obtain more explicit results. Using the results of [12], for $n_{3/2} = 1/2$ polytropes, we have $\xi_1 = (3\pi/4)^{1/2}\Gamma(5/3)/\Gamma(7/6) \simeq 1.49$ and $|\theta'_1| = 2/\sqrt{3} \simeq 1.15$. Therefore, $\omega_{1/2} = 0.349$ and $\mu_* = 0.846$. For $\mu > \mu_* = 0.846$, the normalized energy of a complete white dwarf star (ground state) is

$$\Lambda_{min} = -\frac{3}{7}\left(\frac{\mu_*}{\mu}\right)^{2/3}. \quad (93)$$

5 Conclusion

In this paper, we have studied how the dimension of space affects the nature of phase transitions in the self-gravitating Fermi gas. Since this model has a fundamental interest in astrophysics [6] and statistical mechanics [5], it is important to explore its properties thoroughly even if we sacrifice for practical applications. It is well-known in statistical mechanics that the dimension of space plays a crucial role in the problem of phase transitions. For example, concerning the Ising model, the behaviour in $D = 1$ and $D \geq 2$ is radically different [33]. We have reached a similar conclusion for the self-gravitating Fermi gas. The solution of the problem in $D < 2$ does not yield any phase transition. In $D = 2$, phase transitions appear in the canonical ensemble but not in the microcanonical ensemble. In $D > 2$, phase transitions appear both in microcanonical and canonical ensembles in association with gravitational collapse. The beauty of self-gravitating systems, and other systems with long-range interactions, is their simplicity since the mean-field approximation is exact in any dimension. Therefore, the mean-field theory does *not* predict any phase transition for the self-gravitating Fermi gas in $D = 1$, contrary to the Ising model.

At a more philosophical level, several scientists have examined the role played by the dimension of space in determining the form of the laws of physics. This question goes back to Ptolemy who argues in his treatise *On dimensionality* that no more than three spatial dimensions are possible in Nature. In the 18th century, Kant realizes the deep connection between the inverse square law of gravitation and the existence of three spatial dimensions. In the twentieth century, Ehrenfest [13] argues that planetary orbits, atoms and molecules would be unstable in a space of dimension $D \geq 4$. Other investigations on dimensionality are reviewed in the paper of Barrow [34]. Although we ignored this literature at the beginning, our study clearly enters in this type of investigations. We have found that the self-gravitating Fermi gas possesses a rich structure and displays several characteristic dimensions $D = 2$, $D = 4$, $D = 2(1 + \sqrt{2})$ and $D = 10$. Moreover, as already noted in [12], the dimension $D = 4$ is critical because at that dimension quantum mechanics cannot stabilize matter against gravitational collapse, contrary to the situation in $D = 3$. Interestingly, this result is similar to that of Ehrenfest although it applies to white dwarf stars instead of atoms. The dimension $D = 2$ is also critical as found in [14] and in different domains of physics. Therefore, the dimension of our (macroscopic) universe $D = 3$ plays a very special role regarding the laws of physics (this is illustrated in Fig. 1). Following the far reaching intuition of Kant, we can wonder whether the three space dimensions are a consequence of Newton’s inverse square law, rather than the opposite. We note also that extra-dimensions can appear at the microscale, an idea originating from Kaluza-Klein theory.

This idea took a renaissance in modern theories of grand unification. Our approach shows that already at a simple level, the coupling between Newton's equations (gravitation) and Fermi-Dirac statistics (quantum mechanics) reveals a rich structure as a function of D . Relativistic effects will be considered in a forthcoming paper [29].

Finally, our study can shed light on the mathematical properties of the Vlasov-Poisson system. Indeed, there is a close connexion between collisionless stellar systems and self-gravitating fermions [19, 35, 20, 21]. For example, the fact that the Vlasov equation does not blow up (i.e., experiences gravitational collapse) in $D = 3$ for non singular initial conditions can be related to a sort of exclusion principle, as in quantum mechanics. Due to the Liouville theorem in μ -space, the distribution function must remain smaller than its maximum initial value $f \leq \eta_0$ and this prevents complete collapse [20, 36], unlike for collisional stellar systems [9] described by the Landau-Poisson system. Since quantum mechanics cannot arrest gravitational collapse in $D \geq 4$ (for sufficiently low energies), this suggests that the Vlasov-Poisson system can probably blow up for $D \geq 4$. This remark could be of interest for mathematicians.

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